Exercise 16

Use the *series solution method* to solve the Volterra integral equations:

$$u(x) = x + \int_0^x \tan t\, u(t)\, dt$$

Solution

We seek a series solution for u:

$$u(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots .$$

Substitute this and the Taylor series expansion of $\tan x$,

$$\tan x = x + \frac{1}{3}x^3 + \frac{2}{15}x^5 + \frac{17}{315}x^7 + \frac{62}{2835}x^9 + \cdots ,$$

into the integral equation.

$$\begin{aligned} a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + a_6 x^6 + a_7 x^7 + a_8 x^8 + \cdots \\ = x + \int_0^x \left(t + \frac{1}{3}t^3 + \frac{2}{15}t^5 + \frac{17}{315}t^7 + \cdots\right)(a_0 + a_1 t + a_2 t^2 + a_3 t^3 + \cdots)\, dt \end{aligned}$$

$$\begin{aligned} a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + a_6 x^6 + a_7 x^7 + a_8 x^8 + \cdots \\ = x + \int_0^x t(a_0 + a_1 t + a_2 t^2 + a_3 t^3 + a_4 t^4 + a_5 t^5 + a_6 t^6 + \cdots)\, dt \\ + \frac{1}{3}\int_0^x t^3(a_0 + a_1 t + a_2 t^2 + a_3 t^3 + a_4 t^4 + \cdots)\, dt \\ + \frac{2}{15}\int_0^x t^5(a_0 + a_1 t + a_2 t^2 + \cdots)\, dt + \frac{17}{315}\int_0^x t^7(a_0 + \cdots)\, dt + \cdots \end{aligned}$$

$$\begin{aligned} a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + a_6 x^6 + a_7 x^7 + a_8 x^8 + \cdots \\ = x + \left(\frac{a_0}{2}x^2 + \frac{a_1}{3}x^3 + \frac{a_2}{4}x^4 + \frac{a_3}{5}x^5 + \frac{a_4}{6}x^6 + \frac{a_5}{7}x^7 + \frac{a_6}{8}x^8 + \cdots\right) \\ + \frac{1}{3}\left(\frac{a_0}{4}x^4 + \frac{a_1}{5}x^5 + \frac{a_2}{6}x^6 + \frac{a_3}{7}x^7 + \frac{a_4}{8}x^8 + \cdots\right) \\ + \frac{2}{15}\left(\frac{a_0}{6}x^6 + \frac{a_1}{7}x^7 + \frac{a_2}{8}x^8 + \cdots\right) + \frac{17}{315}\left(\frac{a_0}{8}x^8 + \cdots\right) + \cdots \end{aligned}$$

$$\begin{aligned} a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + a_6 x^6 + a_7 x^7 + a_8 x^8 + \cdots \\ = x + \frac{a_0}{2}x^2 + \frac{a_1}{3}x^3 + \left(\frac{a_2}{4} + \frac{a_0}{12}\right)x^4 + \left(\frac{a_3}{5} + \frac{a_1}{15}\right)x^5 \\ + \left(\frac{a_4}{6} + \frac{a_2}{18} + \frac{a_0}{45}\right)x^6 + \left(\frac{a_5}{7} + \frac{a_3}{21} + \frac{2a_1}{105}\right)x^7 + \left(\frac{a_6}{8} + \frac{a_4}{24} + \frac{a_2}{60} + \frac{17a_0}{2520}\right)x^8 + \cdots \end{aligned}$$

Match the coefficients of the respective powers of x to determine a_i.

$$\begin{aligned}
a_0 &= 0 \\
a_1 &= 1 \\
a_2 &= \frac{a_0}{2} & &\rightarrow & a_2 &= 0 \\
a_3 &= \frac{a_1}{3} & &\rightarrow & a_3 &= \frac{1}{3} \\
a_4 &= \frac{a_2}{4} + \frac{a_0}{12} & &\rightarrow & a_4 &= 0 \\
a_5 &= \frac{a_3}{5} + \frac{a_1}{15} & &\rightarrow & a_5 &= \frac{2}{15} \\
a_6 &= \frac{a_4}{6} + \frac{a_2}{18} + \frac{a_0}{45} & &\rightarrow & a_6 &= 0 \\
a_7 &= \frac{a_5}{7} + \frac{a_3}{21} + \frac{2a_1}{105} & &\rightarrow & a_7 &= \frac{17}{315} \\
a_8 &= \frac{a_6}{8} + \frac{a_4}{24} + \frac{a_2}{60} + \frac{17a_0}{2520} & &\rightarrow & a_8 &= 0 \\
&\vdots & & & &\vdots
\end{aligned}$$

So then

$$\begin{aligned}
u(x) &= x + \frac{1}{3}x^3 + \frac{2}{15}x^5 + \frac{17}{315}x^7 + \cdots \\
&= \tan x.
\end{aligned}$$